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A non-perturbative method for the calculation of Green functions

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Abstract. A new method for the non-perturbative calculation of Green functions in quantum mechanics and quantum field theory is proposed. The method is based on an approximation of the Schwinger–Dyson equation for the generating functional by an exactly soluble equation in functional derivatives. Equations of the leading approximation and the first step are solved for the ϕ_d^4-model. At $d = 1$ (anharmonic oscillator) the ground-state energy is calculated. The renormalization programme is performed for the field theory at $d = 2, 3$. At $d = 4$ the renormalization of the coupling involves a trivialization of the theory.

1. Introduction and general considerations

For many years the construction of non-perturbative approximate solutions has remained an urgent problem of quantum field theory.

In the present work a method for the construction of such approximations is proposed. The method is based on an approximation of the Schwinger–Dyson equation for the generating functional by a simple equation in functional derivatives, which can be solved exactly. This solution is a foundation for the linear iterative scheme. Each step of the scheme consists of solving a closed system of integral equations.

The leading approximation and the first step are investigated by the method for the ϕ_d^4-model. At $d = 1$, when the model corresponds to the anharmonic oscillator, a formula for the ground-state energy is obtained. At $d = 2, 3$ (super-renormalizable field theory) the renormalization of the leading approximation and the first step is performed. At $d = 4$ (strictly renormalizable case) the renormalization of the coupling leads to a non-physical singularity of the amplitude. This is a reflection of the well known triviality problem for the ϕ_4^4-theory in a non-perturbative region.

Consider the theory of a scalar field $\phi(x)$ in Euclidean space ($x \in E_d$) with the action

$$S(\phi) = \int \mathrm{d}x \ \left\{ \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \lambda\phi^4 \right\}. \tag{1}$$

The generating functional of the $2n$-point Green functions can be written as

$$G = \sum_{n=0}^{\infty} G^{2n}\eta^n \tag{2}$$

where $\eta(x, y)$ is a bilocal source. The nth derivative of G over η with the source switched off is the $2n$-point Green function G^{2n}.

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The use of the bilocal source is an essential point of the scheme presented here; therefore we shall consider the theory without spontaneous symmetry breaking with $m^2 > 0$. A brief discussion of the theory with a single source and with spontaneous symmetry breaking is contained at the last section. At $d \geqslant 2$ corresponding counterterms should be included in the action for the cancellation of ultraviolet divergences.

A simple way to obtain the Schwinger–Dyson equation for the generating functional with a bilocal source $\eta(x, y)$ is to use the usual Schwinger–Dyson equation with a single source $j(x)$ in the presence of a bilocal source η:

$$4\lambda \frac{\delta^3 G}{\delta j^3(x)} + (m^2 - \partial^2)\frac{\delta G}{\delta j(x)} = j(x)G + 2\int \mathrm{d}x'\, \eta(x, x')\frac{\delta G}{\delta j(x')}. \tag{3}$$

Differentiating equation (3) over $j(y)$ and using the connection condition

$$\frac{\delta^2 G}{\delta j(x)\delta j(y)} = \frac{\delta G}{\delta \eta(y, x)}$$

after switching off the single source j we obtain the following equation:

$$4\lambda \frac{\delta^2 G}{\delta \eta(y, x)\delta \eta(x, x)} + (m^2 - \partial^2)\frac{\delta G}{\delta \eta(y, x)} - 2\int \mathrm{d}x'\, \eta(x, x')\frac{\delta G}{\delta \eta(y, x')} - \delta(x - y)G = 0. \tag{4}$$

which contains only the source η.

The idea of the iterative scheme presented here is as follows: we shall consider 'an equation with constant coefficients' as a leading approximation, i.e. equation (4) with the next-to-last term omitted. This term contains the source η manifestly. The Green functions are the derivatives of $G(\eta)$ at zero and only the behaviour of G near $\eta = 0$ is essential, therefore such an approximation seems to be acceptable. The equation of the leading approximation will be

$$4\lambda \frac{\delta^2 G_0}{\delta \eta(y, x)\delta \eta(x, x)} + (m^2 - \partial^2)\frac{\delta G_0}{\delta \eta(y, x)} - \delta(x - y)G_0 = 0. \tag{5}$$

The term omitted contains the source and should be treated as a perturbation. Hence, the iteration procedure for the generating functional

$$G = G_0 + G_1 + \cdots + G_n + \cdots \tag{6}$$

consists of the step-by-step solution of the equations

$$4\lambda \frac{\delta^2 G_n}{\delta \eta(y, x)\delta \eta(x, x)} + (m^2 - \partial^2)\frac{\delta G_n}{\delta \eta(y, x)} - \delta(x - y)G_n = 2\int \mathrm{d}x'\, \eta(x, x')\frac{\delta G_{n-1}}{\delta \eta(y, x')}. \tag{7}$$

The solution of the leading approximation equation (5) is a functional:

$$G_0 = \exp \int \mathrm{d}x\, \mathrm{d}y\, \eta(y, x)\, \triangle_0 (x - y) \tag{8}$$

where $\triangle_0$ is a solution of the equation

$$4\lambda\, \triangle_0 (0)\, \triangle_0 (x - y) + (m^2 - \partial^2)\, \triangle_0 (x - y) = \delta(x - y). \tag{9}$$

At $d \geqslant 2$ the quantity $\triangle_0(0)$ must be considered as some regularization.

Equation (9) looks like the self-consistency equation, but differs in the coefficient for λ: in the self-consistency equation the coefficient is three times greater. In this sense equation (9) is more similar to the equation for the propagator in the leading approximation

of the $1/N$-expansion. Certainly the similarity is completely superficial, since the principle of the construction of the approximation scheme is different.

The solution of equation (9) is a free propagator

$$\triangle_0 = \frac{1}{\mu^2 - \partial^2} \tag{10}$$

with the renormalized mass $\mu^2 = m^2 + 4\lambda\,\triangle_0\,(0)$. The quantity $\triangle_0(0)$ is defined from the self-consistency condition.

The propagator is a first derivative of $G(\eta)$ over the source η: $\triangle = \delta G/\delta\eta|_{\eta=0}$. As can easily be seen, it is simply $\triangle_0$ for the leading approximation.

Note that all higher Green functions of the leading approximation starting with the four-point function $G^4 = \delta^2 G/\delta\eta^2|_{\eta=0}$ do not possess the correct connected structure and, correspondingly, the complete bose symmetry. The correct connected structure and other consequences of bose symmetry (e.g., crossing, etc) will be restored in consecutive order in the subsequent steps of the iteration scheme. This is easy to see, for example, by analysing the iteration scheme equations at $\lambda \to 0$. Such a peculiarity of the iteration scheme originates from the bilocal source and is not something exceptional: as is well known, a similar phenomenon also appears in the construction of the $1/N$-expansion in the bilocal source formalism.

The first-step equation for the generating functional G_1 reads

$$4\lambda\frac{\delta^2 G_1}{\delta\eta(y,x)\delta\eta(x,x)} + (m^2 - \partial^2)\frac{\delta G_1}{\delta\eta(y,x)} - \delta(x-y)G_1$$

$$= 2\int \mathrm{d}x'\,\eta(x,x')\,\triangle_0\,(x'-y)G_0 \tag{11}$$

A solution of equation (11) is sought in the form $G_1 = P_1(\eta)G_0$, where $P_1 = \frac{1}{2}F\eta^2 + \triangle_1\eta$. Taking into account the leading approximation, equation (11) gives a system of equations for F and $\triangle_1$:

$$(\mu^2 - \partial_x^2)F\begin{pmatrix} x\,y \\ x'\,y' \end{pmatrix} + 4\lambda F\begin{pmatrix} x\,x \\ x'\,y' \end{pmatrix}\triangle_0\,(x-y)$$

$$= \delta(x-y')\,\triangle_0\,(x'-y) + \delta(x-x')\,\triangle_0\,(y-y') \tag{12}$$

$$(\mu^2 - \partial^2)\,\triangle_1\,(x-y) + 4\lambda\,\triangle_1\,(0)\,\triangle_0\,(x-y) + 4\lambda F\begin{pmatrix} x\,x \\ x\,y \end{pmatrix} = 0. \tag{13}$$

Equation (12) is the linear integral equation for the function F in the momentum space. A solution of the equation is

$$F\begin{pmatrix} x\,y \\ x'\,y' \end{pmatrix} = \triangle_0(x-y')\,\triangle_0\,(x'-y) + \triangle_0(x-x')\,\triangle_0\,(y-y')$$

$$-4\lambda\int \mathrm{d}x_1\,\mathrm{d}y_1\;\triangle_0\,(x-x_1)\,\triangle_0\,(y-x_1)K(x_1-y_1)$$

$$\times\,\triangle_0\,(y_1-x')\,\triangle_0\,(y_1-y') \tag{14}$$

where the kernel K is a solution of the equation

$$K(x-y) = 2\delta(x-y) - 4\lambda\int \mathrm{d}x'\,L(x-x')K(x'-y) \tag{15}$$

and $L(x-y) \equiv \triangle_0^2(x-y)$ is a single loop. Equation (15) can easily be solved in momentum space. Its solution is

$$\tilde{K}(p) = \frac{2}{1+4\lambda\tilde{L}(p)}. \tag{16}$$

Note that the first two terms in equation (14) for F are the missing connected structure of the four-point function to leading approximation. Hence the connected structure of the four-point function is restored at the first iteration step.

To solve equation (13) for $\triangle_1$ is also quite simple. Taking into account the formulae above, the solution can be written as

$$\triangle_1(x-y) = -\int \mathrm{d}x'\,\mathrm{d}y'\ \triangle_0(x-x')\Sigma_r(x'-y')\,\triangle_0(y'-y) \tag{17}$$

where

$$\Sigma_r(x-y) = [4\lambda\,\triangle_1(0) + 8\lambda\,\triangle_0(0)]\delta(x-y) + \Sigma(x-y) \tag{18}$$

$$\Sigma(x-y) = -(4\lambda)^2\,\triangle_0(x-y)\int \mathrm{d}x'\,L(x-x')K(x'-y). \tag{19}$$

The quantity $\triangle_1(0)$ is defined by the self-consistency condition.

At $\lambda \to 0$, as is easy to see, $\triangle = \triangle_0 + \triangle_1 = \triangle^{\mathrm{pert}} + \mathrm{O}(\lambda^2)$, where $\triangle^{\mathrm{pert}}$ is the propagator of the perturbation theory, i.e. at small λ the first-step propagator reproduces correctly the first term of the usual perturbation theory in the coupling.

In the general case, the solution of equation (7) for the nth step of the iteration scheme is

$$G_n = P_n(\eta)G_0 \tag{20}$$

where P_n is a polynomial in η of a degree $2n$. Therefore at the nth step the computation of Green functions reduces to solving a system of $2n$ linear integral equations.

2. The anharmonic oscillator

At $d = 1$ the model with the action (1) describes the quantum mechanical anharmonic oscillator. The parameter m^2 corresponds in the case to a frequency of a harmonic oscillator described by quadratic terms. At $d = 1$ ultraviolet divergences are absent, and the quantities $\triangle_0(0)$ and $\triangle_1(0)$ are finite. Consequently, equations (8)–(19) are applied directly for the computation of Green functions.

Since $\triangle_0(0) = 1/2\mu$ at $d = 1$, the self-consistency condition becomes the equation for a renormalized mass (or, more exactly, for a 'renormalized frequency') μ^2:

$$\mu^2 = m^2 + \frac{2\lambda}{\mu}. \tag{21}$$

Here $\mu = \sqrt{\mu^2}$. The equation has a unique positive solution at all positive m^2 and λ.

To calculate a ground-state energy E one can use the well known formula (see, for example, [1, 2])

$$\frac{\mathrm{d}E}{\mathrm{d}\lambda} = G^4(0,0,0,0). \tag{22}$$

In this formula G^4 is the four-point (or two-particle) function: $G^4 = (1/G)\delta^2 G/\delta\eta^2|_{\eta=0}$. With the formulae above for the Green functions to leading approximation and the first step,

we obtain the following formula for the ground-state energy of the anharmonic oscillator:

$$\frac{\mathrm{d}E}{\mathrm{d}\lambda} = \frac{1}{4\mu^2} + \frac{1}{\mu M}\left(1 - \frac{2\lambda}{\lambda+\mu^3}\left(1 - \frac{2\lambda}{\mu(M+2\mu)^2}\right)\right). \tag{23}$$

Here $M = \sqrt{4\mu^2 + 4\lambda/\mu}$. Integrating the formula with a boundary condition $E|_{\lambda=0} = m/2$ taken into account, one can calculate the ground-state energy for all values of the coupling: $0 \leqslant \lambda < \infty$.

At $\lambda \to 0$, $E = m(\frac{1}{2} + \frac{3}{4}\lambda/m^3 + \mathrm{O}(\lambda^2))$, and the perturbation theory is reproduced up to second order.

At $\lambda \to \infty$, $E = \epsilon_0\lambda^{1/3} + \mathrm{O}(\lambda^{-1/3})$, and $\epsilon_0 = 0.756$. The coefficient ϵ_0 differs by 13% from the exact numerical one $\epsilon_0^{\mathrm{exact}} = 0.668$ (see, for example, [3]).

At $\lambda/m^3 = 0.1$ the result of the calculation with equation (23) differs from the exact numerical one [3] by 0.8%, and at $\lambda/m^3 = 1$ differs by 6.3%.

Consequently, the first-step equation (23) approximates the ground-state energy for all values of λ with an accuracy that varies smoothly from 0 (at $\lambda \to 0$) to 13% (at $\lambda \to \infty$). Comparing these results with the results of other approximate methods, in particular those of variational perturbation theory (VPT) [1, 2] or δ-expansion [4], we can see that this method gives the best results for the intermediate coupling region $\lambda \sim 0.1m^3$. In fact, at $\lambda/m^3 = 0.1$ the accuracy of first step of the calculations is better than that of the fifth step of VPT (see [2, table 6]) and five times higher in comparison with the usual perturbation theory to order $\mathcal{O}(\lambda^5)$. However, for the asymptotic region of strong coupling the accuracy of the given calculations is not so good, and other methods such as VPT give more exact results at $\lambda \to \infty$. It seems likely that a combination of the method proposed with VPT-type methods would give good results for all coupling values.

3. Super-renormalizable theory ($d = 2$ and $d = 3$)

At $d \geqslant 2$ the action (1) should be added by counterterms for the elimination of ultraviolet divergences. First consider the super-renormalizable theory ($d = 2$ and $d = 3$). It is sufficient to add counterterms of mass renormalization $\frac{1}{2}\delta m^2\phi^2$ and wavefunction renormalization $\frac{1}{2}\delta z(\partial_\mu\phi)^2$ in this case. The Schwinger–Dyson equation has the form of equation (4) with the substitution

$$m^2 \to m^2 + \delta m^2 \qquad \partial^2 \to (1+\delta z)\partial^2. \tag{24}$$

There is no need to add a wavefunction renormalization counterterm for the leading approximation, and the equation of the leading approximation will be

$$4\lambda\frac{\delta^2 G_0}{\delta\eta(y,x)\delta\eta(x,x)} + (\delta m_0^2 + m^2 - \partial^2)\frac{\delta G_0}{\delta\eta(y,x)} - \delta(x-y)G_0 = 0. \tag{25}$$

The LHS of the iteration scheme (7) is of the same form as the LHS of (25). At $n \geqslant 1$ the counterterms δm_n^2 and δz_n should be considered as perturbations. Therefore the corresponding terms should be added to the RHS of equation (7). So, the first-step equation will be

$$4\lambda\frac{\delta^2 G_1}{\delta\eta(y,x)\delta\eta(x,x)} + (\delta m_0^2 + m^2 - \partial^2)\frac{\delta G_1}{\delta\eta(y,x)} - \delta(x-y)G_1$$
$$= 2\int \mathrm{d}x'\,\eta(x,x')\frac{\delta G_0}{\delta\eta(y,x')} - \delta m_1^2\frac{\delta G_0}{\delta\eta(y,x)} + \delta z_1\partial^2\frac{\delta G_0}{\delta\eta(y,x)}. \tag{26}$$

The normalization condition on the physical renormalized mass μ^2 gives us a mass renormalization counterterm in the leading approximation

$$\delta m_0^2 = \mu^2 - m^2 - 4\lambda \triangle_0 (0). \tag{27}$$

This counterterm diverges logarithmically at $d = 2$ and linearly at $d = 3$.

For the first step of the iteration equation (12) for F remains unchanged, and its solution is described by the same equations (14)–(16). The equation for $\triangle_1$ changes in correspondence with equation (26). Its solution can be written in the same form (17), but now for Σ_r one gets

$$\Sigma_r = 4\lambda \triangle_1 (0) + \delta m_1^2 - 2\delta m_0^2 - \delta z_1 \partial^2 + \Sigma \tag{28}$$

where Σ is given by equation (19). The normalization conditions

$$\tilde{\Sigma}_r(-\mu^2) = 0 \qquad \tilde{\Sigma}_r'(-\mu^2) = 0 \tag{29}$$

give us the counterterms of the first step

$$\begin{aligned} &\delta z_1 = -\tilde{\Sigma}'(-\mu^2) \\ &\delta m_1^2 = 2\delta m_0^2 - 4\lambda \triangle_1 (0) - \tilde{\Sigma}(-\mu^2) - \mu^2 \tilde{\Sigma}'(-\mu^2). \end{aligned} \tag{30}$$

The renormalized mass operator is

$$\tilde{\Sigma}_r(p^2) = \tilde{\Sigma}(p^2) - \tilde{\Sigma}(-\mu^2) - (p^2 + \mu^2)\tilde{\Sigma}'(-\mu^2) \tag{31}$$

where, in correspondence with (19),

$$\tilde{\Sigma}(p^2) = -(4\lambda)^2 \int \frac{\mathrm{d}q}{(2\pi)^d} \frac{1}{\mu^2 + (p-q)^2} \frac{2\tilde{L}(q^2)}{1 + 4\lambda \tilde{L}(q^2)}. \tag{32}$$

The counterterm δz_1 is finite at $d = 2, 3$. The counterterm δm_1^2 diverges as that of the leading approximation does, namely, logarithmically at $d = 2$ and linearly at $d = 3$. As the simple loop $\tilde{L}(p^2)$ behaves for $p^2 \to \infty$ as $(1/p^2)\log(p^2/\mu^2)$ at $d = 2$ and $1/\sqrt{p^2}$ at $d = 3$, the integral (32) for $\tilde{\Sigma}(p^2)$ converges at $d = 2$ and diverges logarithmically at $d = 3$. Certainly the renormalized mass operator (31) is finite in any case.

A part of the 'redundant' subtractions in equation (31) is completely unclear from the point of view of the usual perturbation theory divergences. This part becomes clear at the strong coupling limit in equations (31), (32). Really, at $\lambda \to \infty$

$$\tilde{\Sigma}(p^2) = \text{constant} + \int \frac{\mathrm{d}q}{(2\pi)^d} \frac{1}{\mu^2 + (p-q)^2} \frac{2}{\tilde{L}(q^2)} + \mathrm{O}\left(\frac{1}{\lambda}\right). \tag{33}$$

The integral in equation (33) diverges *quadratically* at $d = 2, 3$. Hence, it becomes clear that the 'redundant' subtractions conserve the finiteness of the renormalized theory in the strong coupling limit.

4. Strictly renormalizable theory ($d = 4$)

At $d = 4$ besides the renormalizations of the mass and the wavefunction a coupling renormalization is necessary. Therefore simultaneously with the substitution (24) the substitution $\lambda \to \lambda + \delta\lambda$ is also needed in the Schwinger–Dyson equation (4). The leading approximation equation will be

$$4(\lambda + \delta\lambda_0)\frac{\delta^2 G_0}{\delta\eta(y,x)\delta\eta(x,x)} + (\delta m_0^2 + m^2 - \partial^2)\frac{\delta G_0}{\delta\eta(y,x)} - \delta(x-y)G_0 = 0. \tag{34}$$

Due to the presence of the counterterm $\delta\lambda$ the normalization condition on the renormalized mass μ^2 for the leading approximation becomes a connection between counterterms δm_0^2 and $\delta\lambda_0$:

$$\delta m_0^2 + 4(\lambda + \delta\lambda_0)\,\triangle_0(0) = \mu^2 - m^2. \tag{35}$$

As we shall see below, the counterterm $\delta\lambda_0$ (and, consequently, δm_0^2) will be fixed at the *following* step of the iteration scheme.

The first-step equation will be of the form (26) with the substitution $\lambda \to \lambda + \delta\lambda_0$ in the LHS and with an additional term $-4\delta\lambda_1 \cdot \delta^2 G_0/\delta\eta(y,x)\delta\eta(x,x)$ in the RHS. The equation for F will differ from equation (12) only by the substitution $\lambda \to \lambda + \delta\lambda_0$. Therefore the formulae for its solution will also differ from equations (14)–(16) by the same substitution. At $d = 4$ the single-loop integral $\tilde{L}(p^2)$ diverges logarithmically, and renormalization of the coupling is necessary. Let us define a two-particle amplitude—the amputated connected part of the four-point function:

$$A = \triangle_0^{-1}\,\triangle_0^{-1}\,F^{\mathrm{con}}\,\triangle_0^{-1}\,\triangle_0^{-1}. \tag{36}$$

Here a multiplication by $\triangle_0^{-1}$ is understood in the operator sense. F^{con} is the connected part of F. It is easy to see that the amplitude depends only on a variable $p = p_x + p_y$ and has the form

$$\tilde{A}(p^2) = -\frac{8(\lambda + \delta\lambda_0)}{1 + 4(\lambda + \delta\lambda_0)\tilde{L}(p^2)}. \tag{37}$$

Define a renormalized coupling λ_r as a value of the amplitude at a normalization point:

$$\tilde{A}(M^2) = -8\lambda_r = -\frac{8(\lambda + \delta\lambda_0)}{1 + 4(\lambda + \delta\lambda_0)\tilde{L}(M^2)}. \tag{38}$$

From equation (38) one obtains a coupling renormalization counterterm

$$\delta\lambda_0 = -\lambda + \frac{\lambda_r}{1 - 4\lambda_r\tilde{L}(M^2)} \tag{39}$$

and renormalized amplitude

$$\tilde{A}(p^2) = -\frac{8\lambda_r}{1 + 4\lambda_r\tilde{L}_r(p^2; M^2)} \tag{40}$$

where $\tilde{L}_r(p^2; M^2) = \tilde{L}(p^2) - \tilde{L}(M^2)$ is a renormalized loop that possesses a finite limit at the removal of regularization.

Taking the renormalization of the two-particle amplitude in such a manner, one can solve the equation for $\triangle_1$ and renormalize the mass operator in correspondence with the general principle of normalization on the physical mass (see equation (29)). However, in the four-dimensional case one encounters an essential obstacle. At the removal of regularization, $\delta\lambda_0 \to -\lambda$. This is evident from equation (39). Therefore the coefficient $\lambda + \delta\lambda_0$ in the leading approximation (34) vanishes. The same is true for all the subsequent iterations. The theory is then trivialized. One can observe that the expression

$$(\lambda + \delta\lambda_0) \cdot \frac{\delta^2 G}{\delta\eta(y,x)\delta\eta(x,x)} \tag{41}$$

is really an indefinite quantity of the type $0 \cdot \infty$, and the renormalization is, in essence, a definition of the quantity. However, it does not save the situation in this case, since the renormalized amplitude (40) possesses a non-physical singularity in a deep-Euclidean region (it is a well known Landau pole). The unique non-contradictory possibility is a choice

$\lambda_r \to 0$ at the removed regularization. This is again a trivialization of the theory. This trivialization appears almost inevitably in an investigation of ϕ_4^4-theory beyond perturbation theory and is a practically rigorous result (see [5]). Note that in contrast to perturbation theory, which is absolutely non-sensitive to the triviality of the theory, the method proposed already leads to trivialization at the first step.

5. A single source

The method considered above is based essentially on the bilocality of the source. Since the bilocal source produces $2n$-point functions only, the method cannot be applied in its present form to a theory with spontaneous symmetry breaking when

$$\langle 0|\phi|0\rangle \neq 0. \tag{42}$$

For a description of the spontaneous symmetry breaking it is necessary to switch on a single source $j(x)$, i.e. to consider the Schwinger–Dyson equation (3).

Consider a theory with a single source j and with the bilocal source switched off: $\eta = 0$. The Schwinger–Dyson equation for the generating functional $G(j)$ is

$$4\lambda \frac{\delta^3 G}{\delta j^3(x)} + (m^2 - \partial^2)\frac{\delta G}{\delta j(x)} = j(x)G. \tag{43}$$

Let us apply to equation (43) the same idea of approximation by the equation with 'constant' coefficients, i.e. consider as a leading approximation the equation

$$4\lambda \frac{\delta^3 G_0}{\delta j^3(x)} + (m^2 - \partial^2)\frac{\delta G_0}{\delta j(x)} = 0. \tag{44}$$

Then an iteration scheme will be described by the equation

$$4\lambda \frac{\delta^3 G_n}{\delta j^3(x)} + (m^2 - \partial^2)\frac{\delta G_n}{\delta j(x)} = j(x)G_{n-1}. \tag{45}$$

The leading approximation equation (44) has a solution

$$G_0 = \exp\left\{\int \mathrm{d}x\ v(x)j(x)\right\}. \tag{46}$$

Certainly v does not depend on x in a translation-invariant theory. Therefore an equation for v will be

$$4\lambda v^3 + m^2 v = 0. \tag{47}$$

At $m^2 \geqslant 0,\ \lambda > 0$ the equation has the unique real-valued solution $v = 0$, which corresponds to the leading approximation $G_0 = 1$. The iteration scheme (45) based on this leading approximation coincides with the perturbation theory in the coupling—the leading approximation is too simple and does not contain any non-perturbative effects.

At $m^2 < 0$, besides this solution the following real-valued ones exist:

$$v = \pm\sqrt{-\frac{m^2}{4\lambda}} \tag{48}$$

which correspond to spontaneous breaking of a discrete symmetry (P-parity) of ϕ^4-theory. A calculation of the ground-state energy based on equations of the type (22) demonstrates that the state with spontaneous symmetry breaking is energetically preferable and so it is a physical vacuum of the theory at $m^2 < 0$. This is the way of describing the leading non-perturbative effect, i.e. spontaneous symmetry breaking, by the method.

A first-step equation with counterterms will be

$$4\lambda\frac{\delta^3 G_1}{\delta j^3(x)}+(m^2-\partial^2)\frac{\delta G_1}{\delta j(x)}=j(x)G_0-4\delta\lambda_1\frac{\delta^3 G_0}{\delta j^3(x)}-\delta m_1^2\frac{\delta G_0}{\delta j(x)}+\delta z_1\partial^2\frac{\delta G_0}{\delta j(x)}$$

$$=(j(x)-4\delta\lambda_1 v^3-v\delta m_1^2)G_0. \tag{49}$$

A solution of equation (49) should be sought as $G_1 = P_1(j)G_0$, where $P_1(j) = \frac{1}{2}\triangle_1 j^2+\Phi_1 j$. The equation for $\triangle_1$, taking into account the leading approximation equations (46)–(48), has a solution

$$\triangle_1=\frac{1}{\mu^2-\partial^2} \tag{50}$$

where $\mu^2 = -2m^2 > 0$. The reconstruction of the vacuum leads to the corresponding reconstruction of the one-particle spectrum. The whole picture corresponds exactly to a description in the effective potential language, but the notion of the effective potential is not used at all. Subsequent iterations lead to the renormalized perturbation theory over the physical non-symmetrical vacuum. A remarkable feature of the scheme is an absence of symmetry breaking counterterms, even at intermediate steps in the calculation. For the ultraviolet divergences removing the counterterms δm^2, $\delta\lambda$ and δz is sufficient.

In conclusion, note that although at $m^2 > 0$ the values of v in equation (48) are imaginary, the corresponding real-valued solutions of equation (44) exist, for example

$$G_0=\cos\left\{w\int \mathrm{d}x\ j(x)\right\} \tag{51}$$

where $w^2 = m^2/4\lambda$. At the first step of the iteration such a leading approximation gives tachyons and so it is physically unacceptable. It is possible that similar solutions can be useful for an investigation of the problem of spontaneous symmetry breaking in ϕ_2^4-theory with $m^2 > 0$ (see, for example, [6]). Of course, the computational scheme should be modified in that case.

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